Demystifying DT Convolution Sum.

11.1 Graphical Flip/Shift Method

4. add   2. shift   1. flip

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

3. multiply

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

h[n-k] is viewed as function of k with fixed n.

h[n-k] = h[-(k-n)] is a time-reversed and shifted version of impulse response h[k].

Looking at the equation for convolution, we can arrive at the following step-by-step procedure:

1. Choose one signal to be $x[n]$, the other is then $h[n]$; draw them both on the k axis.
2. *FLIP* $h[k]$ about k=0.
3. *SHIFT* flipped version of h to the right by n.
4. *MULTIPLY* $x[k]$ by the flipped/shifted version of $h[k]$ and *ADD* across all values of
5. The summation in step 4 gives you $y[n]$ for only one value of n.
6. Repeat steps 3-5 for all possible values of n.

Convolution of finite length DT Sequences

Checks

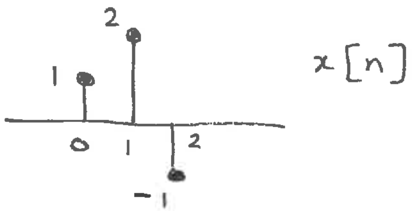
$$y[n] = x[n] * h[n]$$

① $\text{Length}(y[n]) = \text{Length}(x[n]) + \text{Length}(h[n]) - 1$

② $y[n]$ will begin at $\text{Start}(x[n]) + \text{Start}(h[n])$

③ $y[n]$ will finish at $\text{End}(x[n]) + \text{End}(h[n])$

Example



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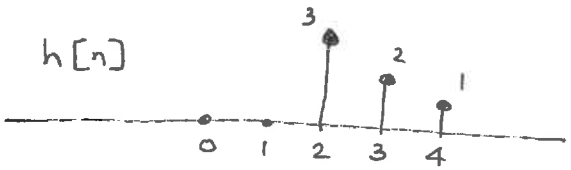

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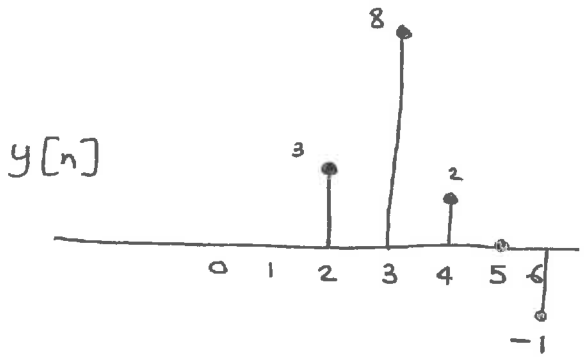


$\text{Length}(x[n]) = 3$

$\text{Start}(x[n]) = 0$

$\text{End}(x[n]) = 2$

$\text{Length}(h[n]) = 3$

$\text{Start}(h[n]) = 2$

$\text{End}(h[n]) = 4$

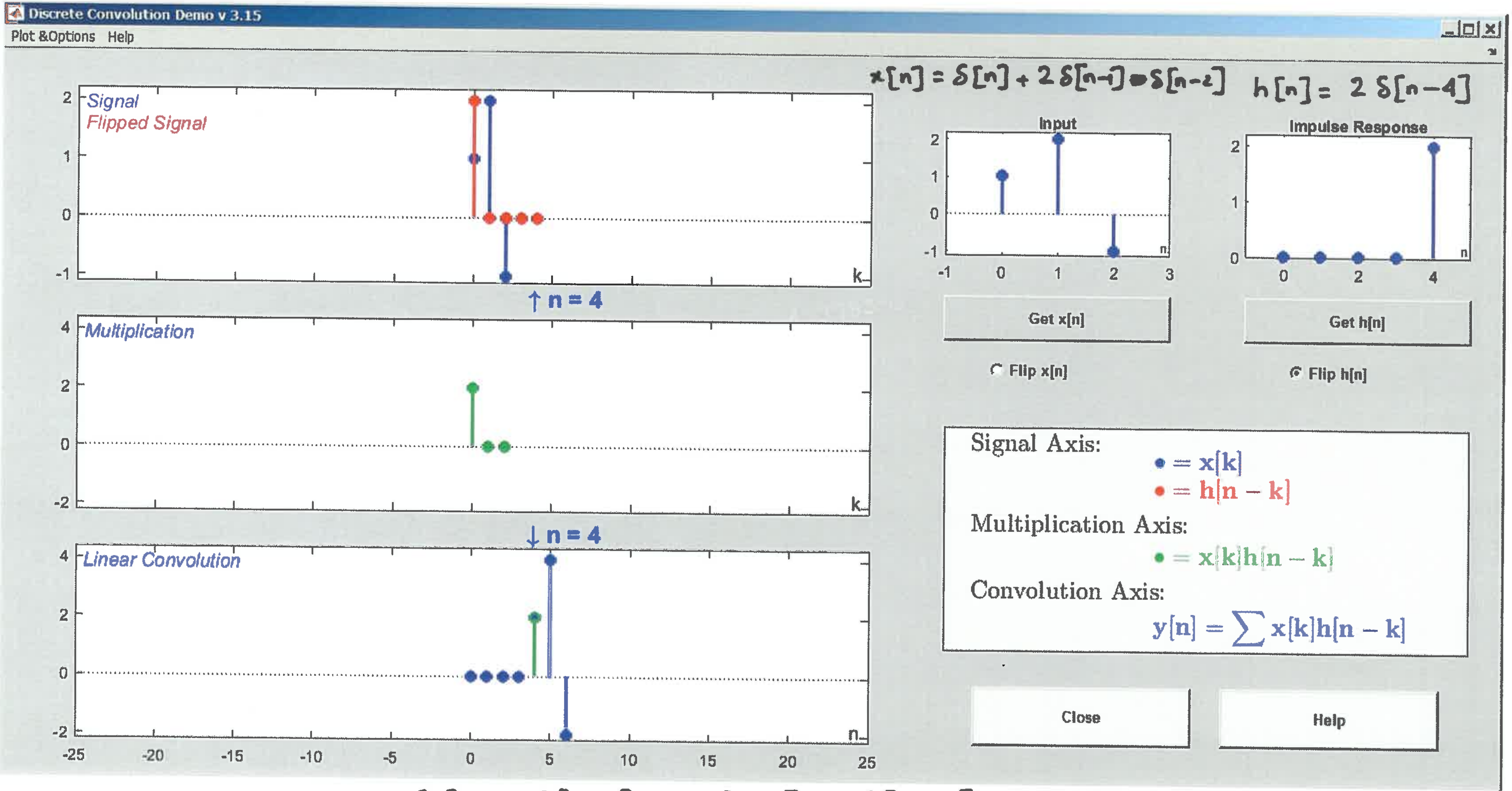
$\text{Length}(y[n]) = 3 + 3 - 1$
$= 5$

$\text{Start}(y[n]) = 0 + 2 = 2$

$\text{End}(y[n]) = 2 + 4 = 6$

Convolution with impulses

$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$ $\quad h[n] = 2\delta[n-4]$



$$y[n] = 2\delta[n-4] + 4\delta[n-5] - 2\delta[n-6]$$

Convolution with Impulses

> Convolution of a signal with a shifted impulse simply shifts the signal to the location of the impulse.

Concept

$\delta[n] * \delta[n-4] = \delta[n-4]$

$\delta[n-1] * \delta[n-4] = \delta[n-4-1] = \delta[n-5]$

replace this n with n−4

$\delta[n-2] * \delta[n-4] = \delta[n-6]$

$(\delta[n] + 2\delta[n-1] - \delta[n-2]) * 2\delta[n-4] = 2\delta[n-4] + 4\delta[n-4-1] - 2\delta[n-4-2]$

$= 2\delta[n-4] + 4\delta[n-5] - 2\delta[n-6]$

Example

$u[n] * \{\delta[n] - \delta[n-1]\}$

$= u[n] * \delta[n] - u[n] * \delta[n-1]$

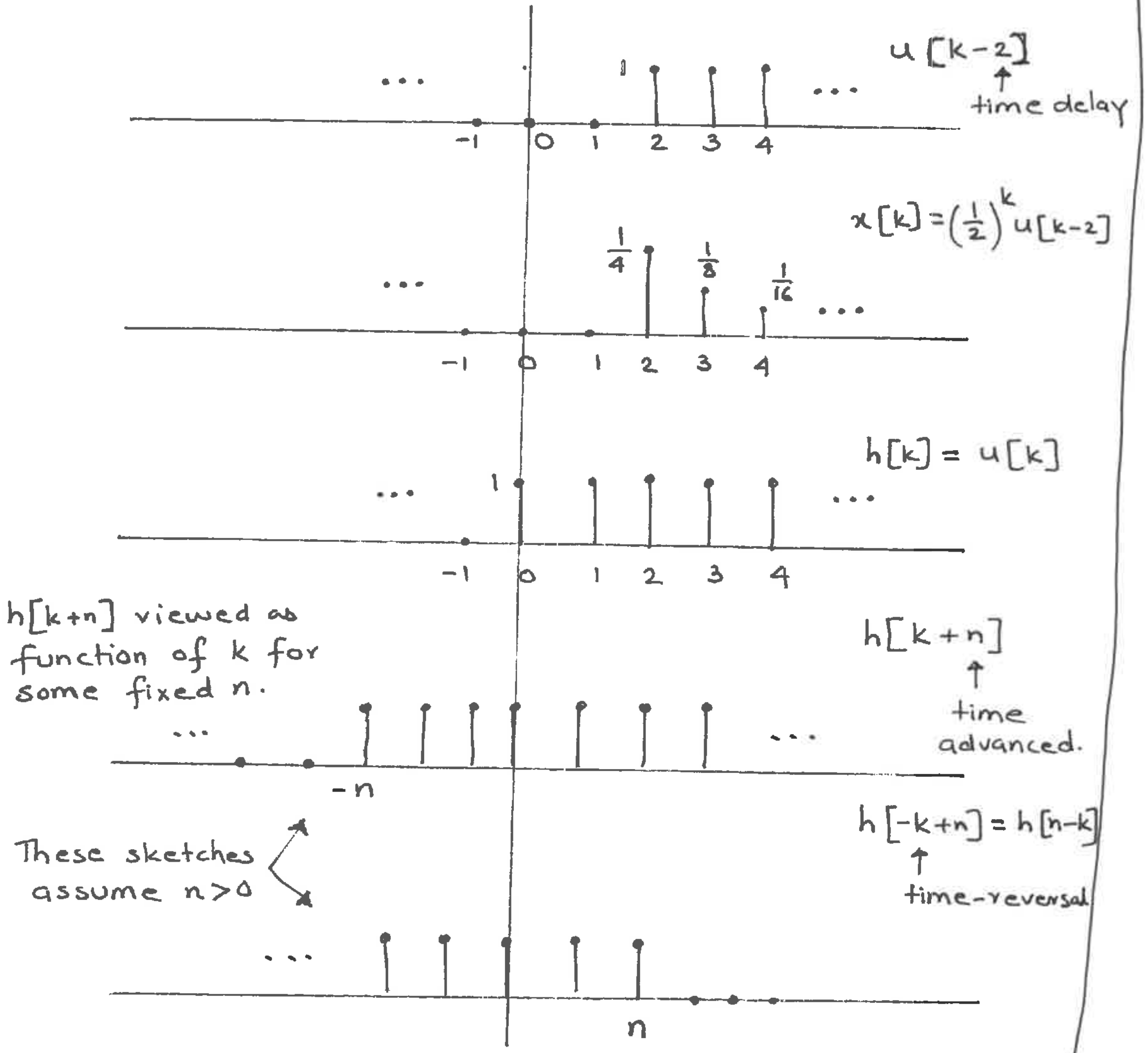
replace n by n−1

$= u[n] - u[n-1]$

Convolution of infinite length DT sequences

Example 1

$$y[n] = \underbrace{\left(\tfrac{1}{2}\right)^n u[n-2]}_{x[n]} * \underbrace{u[n]}_{h[n]}$$



For $n<2$, there is no overlap between the non-zero portions of $x[k]$ and $h[n-k]$. Consequently

$$y[n] = 0.$$

For $n \geq 2$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=2}^{n} \left(\tfrac{1}{2}\right)^k$$

Using formula from sheet

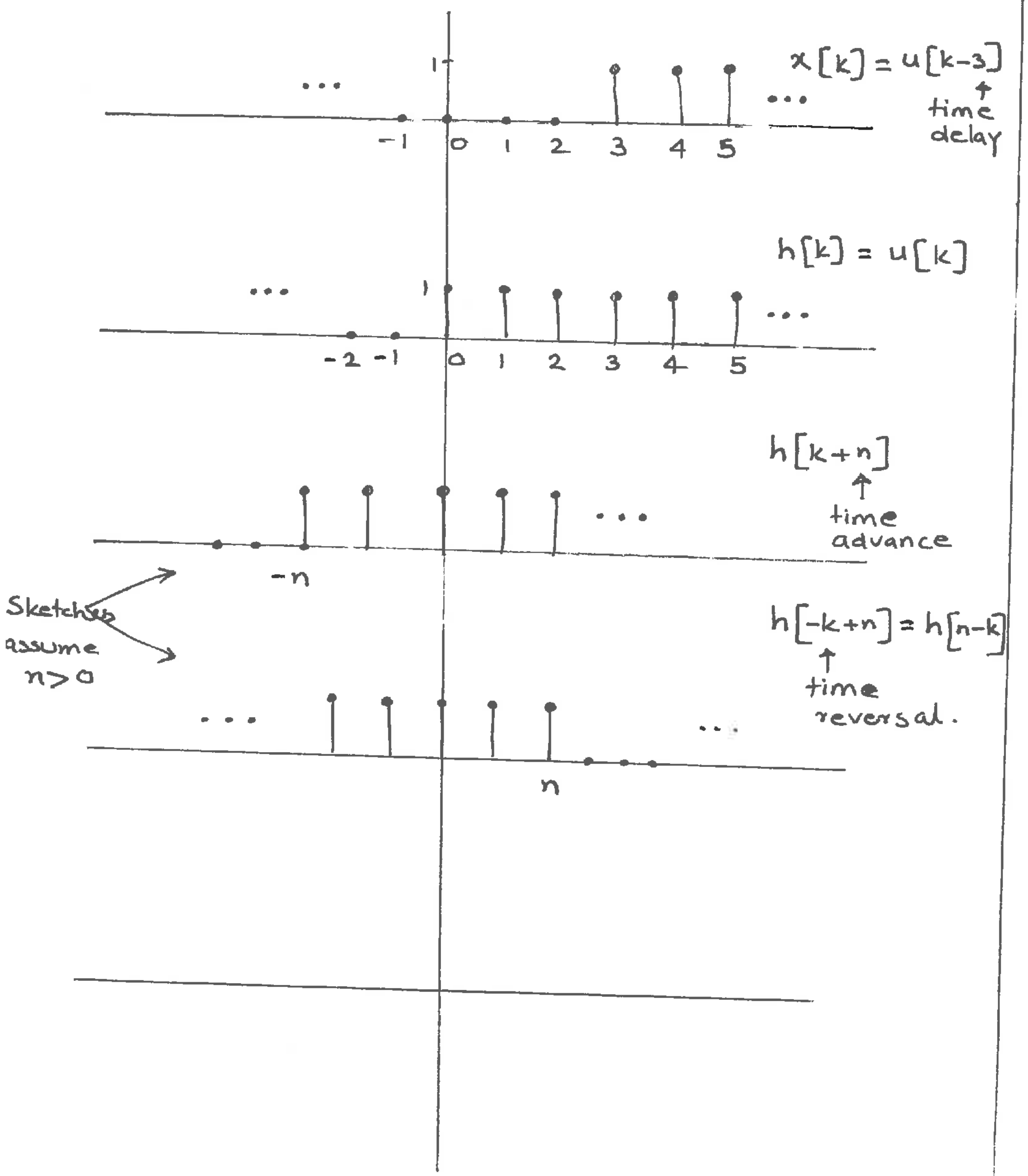
$$y[n] = \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)} - \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)}$$

$$= \frac{1}{2} - \left(\tfrac{1}{2}\right)^{-1}\left(\tfrac{1}{2}\right)^{n+1} = \left(\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)^n.$$

$$y[n] = \begin{cases} 0 & n<2 \\ \frac{1}{2} - \left(\frac{1}{2}\right)^n & n \geq 2 \end{cases}$$

Example 2

$$y[n] = x[n] * h[n] \qquad x[n] = u[n-3], \quad h[n] = u[n]$$



For $n<3$, there is no overlap between non-zero samples of $x[k]$ and $h[n-k]$. Consequently

$$y[n] = 0$$

For $n \geq 3$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=3}^{n} (1)^k \qquad \left\{ \text{Using fact that } 1 = 1^k \text{ for } k>0 \right.$$

$$= n-3+1 = n-2.$$

Alternatively,

$y[3] = 1$, $n=3$

$y[4] = 2$, $n=4$

$y[5] = 3$, $n=5$

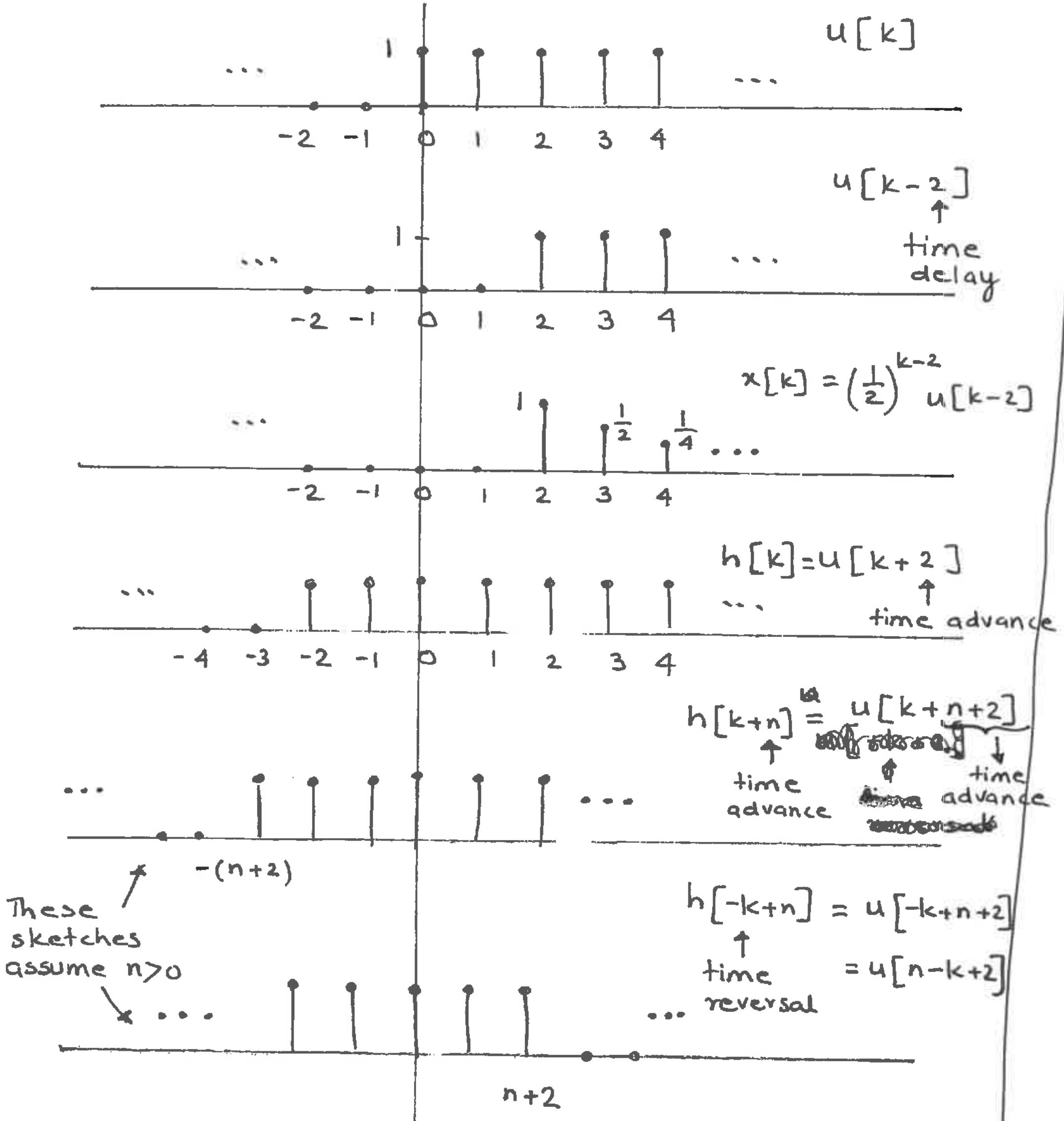
$\therefore\ y[n] = n-2$, $n \geq 3$

Hence $$y[n] = \begin{cases} 0 & n<3 \\ n-2 & n \geq 3 \end{cases}$$

Example 3

$$x[n] = \left(\tfrac{1}{2}\right)^{n-2} u[n-2]$$

$$h[n] = u[n+2]$$



For $n+2 < 2$

$\Rightarrow$ $n < 0$, there is no overlap between the non-zero portions of $x[k]$ and $h[n-k]$. Consequently

$$y[n] = 0$$

For $n+2 \geq 2$

$\Rightarrow$ $n > 0$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=2}^{n+2} \left(\tfrac{1}{2}\right)^{k-2}$$

Changing variable of summation from k to $r = k-2$

$$y[n] = \sum_{r=0}^{n} \left(\tfrac{1}{2}\right)^{r}$$

Using formula sheet

$$y[n] = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = 2\left(1-\left(\tfrac{1}{2}\right)^{n+1}\right)$$

$$= 2\left(1-2^{-n-1}\right)$$

$$= 2-2^{-n}$$

Hence $y[n] = \begin{cases} 0 & n<0 \\ 2-2^{-n} & n \geq 0 \end{cases}$

Convolution Properties

Study from slides 177 – 182.

DT System Properties – Commutative Property

Previously we noted the **Commutative Property**:

$$\boxed{y[n] = x[n] \star h[n] = h[n] \star x[n]}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] \longrightarrow \boxed{x[n]} \longrightarrow y[n]$$

DT System Properties – Commutative Property

- This follows from (change variables to $\ell = n - k$)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

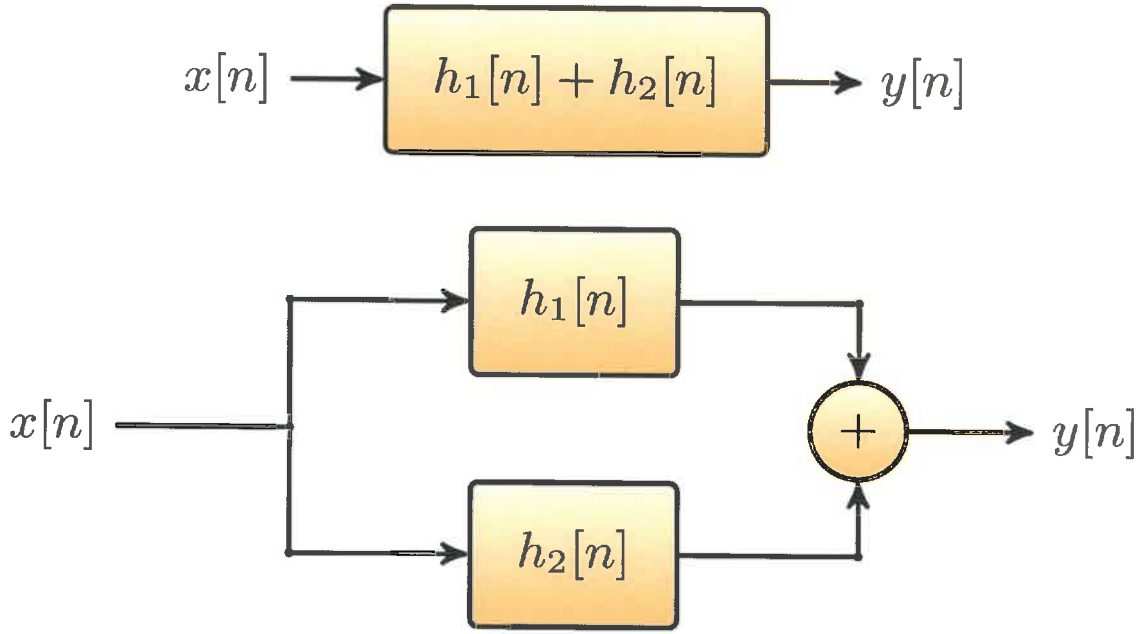
$$= \sum_{\ell=-\infty}^{\infty} x[n-\ell]\, h[\ell] = \sum_{\ell=-\infty}^{\infty} h[\ell]\, x[n-\ell]$$

- Alternatively, if we have two polynomials say $p(x)$ and $q(x)$ then $p(x)q(x) = q(x)p(x)$. Polynomial multiplication is commutative. Convolution is commutative.

DT System Properties – Distributive Property

Consider an input signal $x[n]$ and two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **parallel**, then we have the **Distributive Property**:

$$\boxed{x[n] \star (h_1[n] + h_2[n]) = x[n] \star h_1[n] + x[n] \star h_2[n]}$$

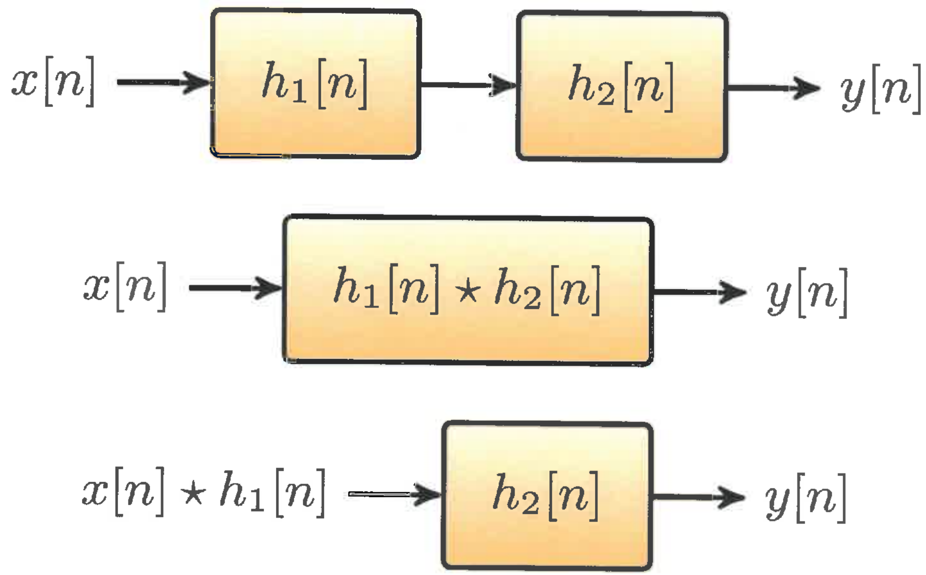


- This implies that we can combine two DT LTI systems in parallel into a single equivalent DT LTI system (by **adding** the pulse responses).

DT System Properties – Associative Property

Consider an input signal $x[n]$ to two DT LTI Systems $h_1[n]$ and $h_2[n]$, in **cascade**, then we have the **Associative Property**:

$$\boxed{x[n] \star (h_1[n] \star h_2[n]) = (x[n] \star h_1[n]) \star h_2[n]}$$



- This implies that we can combine two DT LTI systems in series into a single equivalent DT LTI system (by **convolving** the pulse responses).